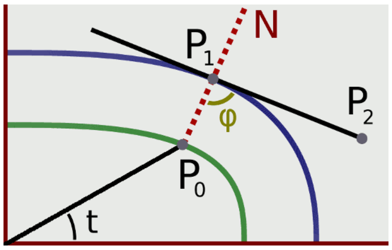
Tangent Line to a Hyperellipse Coated With Layer

Parametric equation of ¼ hyperellipse coated with layer of thickness r_1, equation of normal vector:

$$\begin{aligned}
\vec{N} &= (N_x, N_y) = \left(\frac{\cos^{n-1}t}{a^n}, \frac{\sin^{n-1}t}{b^n}\right), \\
x &= \cos(t)\left(\frac{\cos^n t}{a^n}+\frac{\sin^n t}{b^n}\right)^{-1/n} + r_1 N_x \left(\frac{\cos^{2n-2}t}{a^{2n}}+\frac{\sin^{2n-2}t}{b^{2n}}\right)^{-1/2}, \\
y &= \sin(t)\left(\frac{\cos^n t}{a^n}+\frac{\sin^n t}{b^n}\right)^{-1/n} + r_1 N_y \left(\frac{\cos^{2n-2}t}{a^{2n}}+\frac{\sin^{2n-2}t}{b^{2n}}\right)^{-1/2}, \quad n\geqslant 2,\ t\in[0,\pi/2].
\end{aligned} \tag{1}$$

Find tangent line passing through point P_2.



Vector difference $\vec{P}_1-\vec{P}_2$ must be perpendicular to $\vec{N}$ which is a normal vector at point P_0. For that, cosine of the angle between vectors (equals to dot product of vectors of length 1) must be zero:

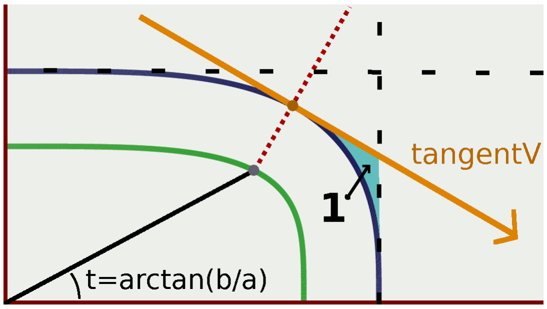
$$\cos\varphi = \frac{(\vec{P}_1-\vec{P}_2)\cdot\vec{N}}{\|\vec{P}_1-\vec{P}_2\|\cdot\|\vec{N}\|} = \frac{(P_{1x}-P_{2x})N_x+(P_{1y}-P_{2y})N_y}{\sqrt{N_x^2+N_y^2}\sqrt{(P_{1x}-P_{2x})^2+(P_{1y}-P_{2y})^2}} = 0. \tag{2}$$

Cancellation out $\|\vec{P}_1-\vec{P}_2\|\cdot\|\vec{N}\|$ is not performed, to allow root-finding algorithm use absolute error bound expressed in $\cos\varphi$. Plugged are $\vec{N}, x, y$ from equations (1):

$$\cos\varphi = \left(r_1+\left(\frac{\cos^{2n-2}t}{a^{2n}}+\frac{\sin^{2n-2}t}{b^{2n}}\right)^{-1/2}\left(\frac{\cos^n t}{a^n}+\frac{\sin^n t}{b^n}\right)^{-1/n+1} - P_{2x}\frac{\cos^{n-1}t}{a^n} - P_{2y}\frac{\sin^{n-1}t}{b^n}\right) \Big/ \sqrt{(x-P_{2x})^2+(y-P_{2y})^2}. \tag{3}$$

Function properties

- If P_2 is inside the closed curve then tangent lines do not exist. In other cases 2 tangent lines exist: to the left and to the right of the ray from the origin through P_2.
- If required P_2 is translated such that P_1 appears in the quadrant where $P_{1x}\geqslant 0, P_{1y}\geqslant 0$. After the computation P_1 is translated back accordingly.
- tangentV is tangent vector at the “corner” of a hyperellipse directed towards X+. It is determined if P_2 is to the left or to the right of tangentV and subsequently it is determined the interval where P_1 is located: $t\in[0, \arctan(b/a)] \vee t\in[\arctan(b/a), \pi/2]$.
- If P_2 is to the right of tangentV ($\vec{P}_2\cdot\overrightarrow{\text{tangentV}}\geqslant 0$) and inside bounding box ($P_{2x}\leqslant a+r_1 \wedge P_{2y}\leqslant b+r_1$) then two cases possible: P_2 is inside the hyperellipse or 2 tangent lines exist on the interval determined above. Then algorithm for finding point on a hyperellipse closest to P_2 is executed. If P_2 is not inside then closest point parameter is used to divide interval.
- Function is continuously differentiable on the interval $t\in[0,\pi/2]$. Derivative value is computed along with function value using automated software process.



Filled-in, designated number **1** is the area such that if P_2 is inside the area then 2 tangent lines exist on the interval [0, arctan(b/a)].

Applied is Newton-Raphson method combined with bisection as described in “Distance Hyperellipse to Point”. Initial guess t_0 is set proportional to distance P_2 to tangentV and distance P_2 to axis X or axis Y.

It was measured average number of evaluations of function (3) and its derivative to reach absolute error below 10^{-12}. It depends on hyperellipse parameters n, a, b, r_1 and distance to the point. The result is 4.2 to 9.7 evaluations.